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Dimensions of irreducible representations of the classical Lie groups

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Abstract. Formulae are derived expressing the dimensions of each irreducible representation of each of the classical Lie groups of transformations in an N-dimensional space as a factored polynomial in N divided by a product of hook length factors.

1. Introduction

Weyl (1925, 1926) derived explicit formulae for the dimensions of each of the irreducible representations of the classical semi-simple Lie groups SU(N), SO(N) and Sp(N). However, these formulae do not make manifest the N dependence of the results. The first formula which did do this in the case of the irreducible tensor representations of $U(N)$ is due to Robinson (1961 p 60). This formula expressed the dimension of the irreducible representation of U(N) with character $\{\boldsymbol{\lambda}\}$ in the form of a numerator consisting of a factored polynomial in N and a denominator $H(\boldsymbol{\lambda})$ which is a product of hook length factors. This was generalised to cover the characters $\{\bar{\boldsymbol{\nu}}; \boldsymbol{\mu}\}$ of U(N) for which a dimensionality formula was derived by two distinct methods (El Samra 1970, Jahn and El Samra 1970 (unpublished), King 1970), but in both cases the derivation was unattractively lengthy. Similarly a further generalisation was made to the case of irreducible tensor representations of O(N) and Sp(N) having characters $[\boldsymbol{\lambda}]$ and $\langle\boldsymbol{\lambda}\rangle$ (Abramsky *et al* 1973). In these cases whilst the denominators were of the required form, namely $H(\boldsymbol{\lambda})$, the numerators were expressed in a rather cumbersome manner.

In this paper two important lemmas due to Cauchy (Muir 1906) and Frobenius (1903) are stated in the following section, where their use in deriving Robinson's hook length formula from Weyl's dimensionality formula is presented in detail. The derivation makes use of the reduced determinantal form of the characters $\{\boldsymbol{\lambda}\}$ given by Littlewood (1940 p 112) and discussed in the preceding paper (El Samra and King 1979).

In § 3 exactly the same method is used to derive dimensionality formulae appropriate to the tensor representations of O(N) and Sp(N). This time the numerators are expressed very succinctly, making the results easier to use. The same procedure is followed in § 5 in the case of the mixed tensor irreducible representations of U(N)

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having character $\{\tilde{\nu}; \boldsymbol{\mu}\}$, whilst in § 6 the formulae for the spinor characters of $O(N)$ and $SO(N)$ given in the preceding paper are exploited to give the dimensions of the remaining irreducible representations of the classical groups.

2. Robinson's hook length formula

The dimension $D_N\{\boldsymbol{\lambda}\}$ of the irreducible representation $\{\boldsymbol{\lambda}\}$ of $U(N)$, which remains irreducible on restriction to $SU(N)$, has been given by Weyl (1925 § 6 (41)) in the form

$$D_N\{\boldsymbol{\lambda}\} = \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j - i + j)/(-i+j). \tag{2.1}$$

To determine explicitly the dependence on N it is convenient to make use of the reduced determinantal form of $\{\boldsymbol{\lambda}\}$ given by Littlewood (1940 p 112) and rederived by Foulkes (1951). This is best stated and proved using the Frobenius notation for partitions, which is such that

$$(\boldsymbol{\lambda}) = \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} a_1 a_2 \ldots a_r \\ b_1 b_2 \ldots b_r \end{pmatrix} \tag{2.2}$$

with $a_i = \lambda_i - i$ and $b_j = \tilde{\lambda}_j - j$ for $i, j = 1, 2, \ldots$, so that $a_i > a_{i+1}$ and $b_j > b_{j+1}$ for all i and j, whilst r is defined by the conditions $a_i \geq 0$ and $b_j \geq 0$ for $i, j = 1, 2, \ldots, r$ and $a_i < 0$ and $b_j < 0$ for $i, j = r+1, r+2, \ldots$.

With this notation the reduced determinantal expansion takes the form

$$\{\boldsymbol{\lambda}\} = \left| \begin{Bmatrix} a_i \\ b_j \end{Bmatrix} \right| \tag{2.3}$$

where $i, j = 1, 2, \ldots, r$, so that

$$D_N\{\boldsymbol{\lambda}\} = \left| D_N \begin{Bmatrix} a_i \\ b_j \end{Bmatrix} \right|. \tag{2.4}$$

Applying (2.1) to the case $(\boldsymbol{\lambda}) = (a+1, 1^b) = \binom{a}{b}$ and separating out those terms for which $i = 1$ gives

$$D_N\{a+1, 1^b\} = \frac{(N+a)!}{a!(N-1)!(a+b+1)} D_{N-1}\{1^b\}. \tag{2.5}$$

From this follows, as a special case, the recurrence relation

$$D_N\{1^{b+1}\} = \frac{N}{(b+1)} D_{N-1}\{1^b\}. \tag{2.6}$$

Since $D_M\{0\} = 1$ for $M \geq 1$ and $D_1\{1\} = 1$ the solution of this recurrence relation is

$$D_N\{1^m\} = N!/m!(N-m)! \qquad \text{for } 0 \leq m \leq N. \tag{2.7}$$

Substitution into (2.5) then yields the result needed to exploit (2.4), namely

$$D_N \begin{Bmatrix} a \\ b \end{Bmatrix} = (N+a)!/(N-b-1)!a!b!(a+b+1). \tag{2.8}$$

Using this in (2.4) gives, after extracting the various factors common to each element in any given row or column, the result

$$D_N\begin{Bmatrix} a \\ b \end{Bmatrix} = \prod_{i=1}^{r} \frac{(N+a_i)!}{(N-b_i-1)!a_i!b_i!} \left| \frac{1}{(a_i+b_j+1)} \right|. \tag{2.9}$$

To evaluate this and many similar determinants below, it is only necessary to recall Cauchy's lemma (Muir 1906 p 342). This states that

$$\left| \frac{1}{(x_i-y_j)} \right| = (-1)^{r(r-1)/2} \prod_{1\leqslant i<j\leqslant r} (x_i-x_j)(y_i-y_j) \left(\prod_{i=1}^{r} \prod_{j=1}^{r} (x_i-y_j) \right)^{-1}. \tag{2.10}$$

Its proof depends on noticing that the left-hand side vanishes if $x_i = x_j$ or $y_i = y_j$ for any pair (i, j) with $i \neq j$. After extracting the appropriate factors $(x_i - x_j)$ and $(y_i - y_j)$, and the common denominator, consideration of the degree of the numerator as a multinomial in x_i and y_j leaves only the numerical factor $(-1)^{r(r-1)/2}$ to be found. This may be obtained by considering, for example, the leading term in the expansion.

Taking $x_i = a_i$ and $y_j = -1-b_j$ in this lemma for $i, j = 1, 2, \ldots, r$ then gives

$$D_n\begin{Bmatrix} a \\ b \end{Bmatrix} = \prod_{i=1}^{r} \left\{ \frac{(N+a_i)!}{(N-b_i-1)!} \left[\frac{1}{a_i!} \prod_{j=i+1}^{r} (a_i-a_j) \right] \right.$$
$$\left. \times \left[\frac{1}{b_i!} \prod_{j=i+1}^{r} (b_i-b_j) \right] \right\} \left(\prod_{i=1}^{r} \prod_{j=1}^{r} (a_i+b_j+1) \right)^{-1}. \tag{2.11}$$

However, there exists another important lemma due to Frobenius (1903 § 5) which states that the sets $\{a_i: i = 1, 2, \ldots\}$ and $\{-1+b_j: j = 1, 2, \ldots\}$ have no element in common. This was proved in the preceding paper (El Samra and King 1979). The implication of this lemma here is that for each fixed value of i in the range $1 \leqslant i \leqslant r$ the particular sets

$S_a = \{(a_i - a_j): j = i+1, i+2, \ldots, r\}$ and $S_b = \{(a_i + b_j + 1): j = r+1, r+2, \ldots, \lambda_i\}$

are disjoint. Moreover,

$$a_i \geqslant a_i - a_r > a_i - a_{r-1} > \ldots > a_i - a_{i+1} \geqslant 1$$

and

$$a_i \geqslant a_i + b_{r+1} + 1 > a_i + b_{r+2} + 1 > \ldots > a_i + b_{\lambda_i} + 1 = a_i + \tilde{\lambda}_{\lambda_i} - \lambda_i + 1 \geqslant a_i + i - \lambda_i + 1 = 1$$

where use has been made of the fact that $\tilde{\lambda}_{\lambda_i} \geqslant i$. This merely corresponds to the observation that in the Young diagram specified by the partition $\boldsymbol{\lambda}$ the length of the λ_ith column is at least i for all $i \leqslant r$. Thus S_a and S_b are both subsets of S where

$$S = \{(a_i - j): j = 0, 1, 2, \ldots, a_i - 1\}.$$

However, as already stated, S_a and S_b are disjoint and the total number of elements of S_a and S_b is precisely $(r-i)+(\lambda_i - r) = a_i$, which is the number of elements of S. Necessarily, therefore, S is the disjoint union of S_a and S_b, from which it follows that

$$\frac{1}{a_i!} \prod_{j=i+1}^{r} (a_i - a_j) = \left(\prod_{j=r+1}^{\lambda_i} (a_i + b_j + 1) \right)^{-1}. \tag{2.12}$$

Similarly

$$\frac{1}{b_i!} \prod_{j=i+1}^{r} (b_i - b_j) = \left(\prod_{j=r+1}^{\tilde{\lambda}_i} (a_j + b_i + 1) \right)^{-1}. \tag{2.13}$$

Substituting (2.12) and (2.13) in (2.11) gives

$$D_N\begin{Bmatrix} a \\ b \end{Bmatrix} = \prod_{i=1}^{r} \frac{(N+a_i)!}{(N-b_i-1)!} \Big/ H\begin{pmatrix} a \\ b \end{pmatrix} \tag{2.14}$$

where

$$H\begin{pmatrix} a \\ b \end{pmatrix} = \prod_{(i,j)}^{\lambda} (a_i + b_j + 1) \tag{2.15}$$

with the product taken over all pairs (i, j) specifying the position of a box in the Young diagram corresponding to the partition $(\boldsymbol{\lambda}) = \binom{a}{b}$.

The formula (2.14) may then be rewritten to yield the famous dimensionality formula of Robinson (1961 p 60):

$$D_N\{\boldsymbol{\lambda}\} = \prod_{(i,j)}^{\lambda} (N-i+j) \left(\prod_{(i,j)}^{\lambda} (\lambda_i - j + \tilde{\lambda}_j - i + 1) \right)^{-1} \tag{2.16}$$

in which the denominator is the product of all the hook length factors of the boxes in the Young diagram specified by $\boldsymbol{\lambda}$. The numerator giving the N dependence of the result is given by a factored polynomial in which each factor is associated with a box of the Young diagram in an obvious way. Thus for example

$$D_N\{4\,3^2\,2\,1\} = \begin{matrix} (N) & (N+1) & (N+2) & (N+3) \\ (N-1) & (N) & (N+1) & \\ (N-2) & (N-1) & (N) & \\ (N-3) & (N-2) & & \\ (N-4) & & & \end{matrix} \Bigg/ \begin{matrix} 8 & 6 & 4 & 1 \\ 6 & 4 & 2 & \\ 5 & 3 & 1 & \\ 3 & 1 & & \\ 1. & & & \end{matrix}$$

3. Orthogonal and symplectic groups

The derivation of the previous section has been given in full because the steps followed are just those necessary for the derivation of similar results for the tensor representations of the orthogonal and symplectic groups. The starting point is once more one or other of the formulae due to Weyl (1926 § 3 (24), § 5 (35), (35′)). These give the dimensions of the irreducible characters $[\boldsymbol{\lambda}]$ and $\langle\boldsymbol{\lambda}\rangle$ of SO(N) and Sp(N) and are such that

$$D_{2k}[\boldsymbol{\lambda}] = \prod_{1\leq i<j\leq k} \frac{(\lambda_i - \lambda_j - i + j)(\lambda_i + \lambda_j - i - j + 2k)}{(-i+j)(-i-j+2k)}, \tag{3.1}$$

$$D_{2k+1}[\boldsymbol{\lambda}] = \prod_{1\leq i<j\leq k} \frac{(\lambda_i - \lambda_j - i + j)}{(-i+j)} \prod_{1\leq i\leq j\leq k} \frac{(\lambda_i + \lambda_j - i - j + 2k + 1)}{(-i-j+2k+1)}, \tag{3.2}$$

and

$$D_{2k}\langle\boldsymbol{\lambda}\rangle = \prod_{1\leq i<j\leq k} \frac{(\lambda_i - \lambda_j - i + j)}{(-i+j)} \prod_{1\leq i\leq j\leq k} \frac{(\lambda_i + \lambda_j - i - j + 2k + 2)}{(-i-j+2k+2)}. \tag{3.3}$$

Applying these formulae to the cases for which $(\boldsymbol{\lambda}) = \binom{a}{b} = (a+1, 1^b)$ with $b+1 \leq k$ gives

$$D_{2k}[a+1, 1^b] = \frac{(a+2k-1)!(a+k)}{a!(2k-3)!(k-1)(a+b+1)(a-b+2k-1)} D_{2k-2}[1^b],$$
$$D_{2k+1}[a+1, 1^b] = \frac{(2a+2k+1)!}{a!(2k-1)!(a+b+1)(a-b+2k)} D_{2k-1}[1^b] \tag{3.4}$$

and

$$D_{2k}\langle a+1, 1^b\rangle = \frac{(a+2k+1)!}{a!(2k-1)!(a+b+1)(a-b+2k+1)} D_{2k-2}\langle 1^b\rangle, \tag{3.5}$$

and hence the recurrence relations

$$D_{2k}[1^{b+1}] = \frac{2k(2k-1)}{(b+1)(2k-b-1)} D_{2k-2}[1^b], \tag{3.6}$$

$$D_{2k+1}[1^{b+1}] = \frac{(2k+1)2k}{(b+1)(2k-b)} D_{2k-1}[1^b] \tag{3.7}$$

and

$$D_{2k}\langle 1^{b+1}\rangle = \frac{(2k+1)2k}{(b+1)(2k-b+1)} D_{2k-2}\langle 1^b\rangle. \tag{3.8}$$

It should be noted that the first of these, equation (3.6), is not valid in the special case for which $b+1 = k$. In this case it is important to recognise that the character $[1^{b+1}]$ should more properly be denoted by $[1^k]_+$. This is because Weyl's formula (3.1), from which (3.6) is derived, applies to irreducible representations of SO($2k$). The character $[1^k]$ is the character of a reducible representation of SO($2k$). The two irreducible constituents are of the same dimension and have characters $[1^k]_\pm$. It is easy to see directly from (3.1) that

$$D_{2k}[1^k] = 2D_{2k}[1^k]_\pm = (2k)!/k!k!. \tag{3.9}$$

The solutions to equations (3.6), (3.7) and (3.8) following from the fact that $D_M[0] = D_M\langle 0\rangle = 1$ for $M \geq 1$ and $D_2\langle 1\rangle = 2$ take the form

$$D_{2k}[1^m] = (2k)!/m!(2k-m)!, \tag{3.10}$$

$$D_{2k+1}[1^m] = (2k+1)!/m!(2k+1-m)! \tag{3.11}$$

and

$$D_{2k}\langle 1^m\rangle = (2k+2-2m)(2k+1)!/m!(2k+2-m)!. \tag{3.12}$$

Substitution into equations (3.4) and (3.5) then yields, for the dimensions of irreducible characters of O(N) and Sp(N) associated with Young diagrams consisting of single hooks, the formulae

$$D_N\begin{bmatrix} a \\ b \end{bmatrix} = \frac{(N+a-1)!(N+2a)}{(N-b-2)!(N+a-b-1)a!b!(a+b+1)} \tag{3.13}$$

and

$$D_N\left\langle \begin{matrix} a \\ b \end{matrix} \right\rangle = \frac{(N+a+1)!(N-2b)}{(N-b)!(N+a-b+1)a!b!(a+b+1)}. \tag{3.14}$$

These are just what is required for exploiting the reduced determinantal expansions (Abramsky *et al* 1973, El Samra and King 1979)

$$\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = \left| \begin{bmatrix} a_i \\ b_j \end{bmatrix} \right| \qquad \text{and} \qquad \left\langle \begin{matrix} \boldsymbol{a} \\ \boldsymbol{b} \end{matrix} \right\rangle = \left| \left\langle \begin{matrix} a_i \\ a_j \end{matrix} \right\rangle \right|. \tag{3.15), (3.16}$$

The analogues of equation (2.9) then take the form

$$D_N \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = \prod_{i=1}^{r} \frac{(N+a_i-1)!(N+2a_i)}{(N-b_i-2)!a_i!b_i!} \left| \frac{1}{(N+a_i-b_j-1)(a_i+b_j+1)} \right| \tag{3.17}$$

and

$$D_N \left\langle \begin{matrix} \boldsymbol{a} \\ \boldsymbol{b} \end{matrix} \right\rangle = \prod_{i=1}^{r} \frac{(N+a_i+1)!(N-2b_i)}{(N-b_i)!a_i!b_i!} \left| \frac{1}{(N+a_i-b_j+1)(a_i+b_j+1)} \right|. \tag{3.18}$$

The denominators of the ijth terms of the determinants can be written in the forms

$$(\tfrac{1}{2}N+a_i)^2-(\tfrac{1}{2}N-b_j-1)^2$$

and

$$(\tfrac{1}{2}N+a_i+1)^2-(\tfrac{1}{2}N-b_j)^2$$

which are directly amenable to the use of Cauchy's lemma. Its application yields

$$D_N \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = \prod_{i=1}^{r} \frac{(N+a_i-1)!(N+2a_i)}{(N-b_i-2)!a_i!b_i!} \prod_{1\leq i<j\leq r} (a_i-a_j)(N+a_i+a_j)(b_i-b_j)(N-b_i-b_j-2)$$
$$\times \left(\prod_{i=1}^{r} \prod_{j=1}^{r} (N+a_i-b_j-1)(a_i+b_j+1) \right)^{-1} \tag{3.19}$$

and

$$D_N \left\langle \begin{matrix} \boldsymbol{a} \\ \boldsymbol{b} \end{matrix} \right\rangle = \prod_{i=1}^{r} \frac{(N+a_i+1)!(N-2b_i)}{(N-b_i)!a_i!b_i!} \prod_{1\leq i<j\leq r} (a_i-a_j)(N+a_i+a_j+2)(b_i-b_j)(N-b_i-b_j)$$
$$\times \left(\prod_{i=1}^{r} \prod_{j=1}^{r} (N+a_i-b_j+1)(a_i+b_j+1) \right)^{-1} \tag{3.20}$$

It is a straightforward task to extract the factor $H(\begin{smallmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{smallmatrix})$ in each denominator of these expressions as in the derivation of (2.14), and the N-dependent factors may be rewritten to give

$$D_N \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = \prod_{i=1}^{r} \left(\frac{(N+a_i-1)!}{(N-b_i-2)!} \prod_{j=i}^{r} \frac{(N+a_i+a_j)}{(N+a_i-b_j-1)} \prod_{j=i+1}^{r} \frac{(N-b_i-b_j+2)}{(N-b_i+a_j-1)} \right) \Big/ H\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \tag{3.21}$$

and

$$D_N \left\langle \begin{matrix} \boldsymbol{a} \\ \boldsymbol{b} \end{matrix} \right\rangle = \prod_{i=1}^{r} \left(\frac{(N+a_i+1)!}{(N-b_i)!} \prod_{j=i+1}^{r} \frac{(N+a_i+a_j+2)}{(N+a_i-b_j+1)} \prod_{j=i}^{r} \frac{(N-b_i-b_j)}{(N-b_i+a_j+1)} \right) \Big/ H\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}. \tag{3.22}$$

These are essentially the results given previously (Abramsky *et al* 1973) in forms which make it clear that there exists a pattern of cancellation of the various denominator factors depending on N dictated by the hook structure of the Young diagram specified by $\boldsymbol{\lambda}$. However, in this earlier paper the algebraic form of the result of carrying out these cancellations was not obtained. The key to overcoming this problem is once again Frobenius' lemma which may be used to show that the following sets are disjoint:

$$\{(N+a_i-b_j-1);\ j=i,i+1,\ldots,r\} \qquad \{(N+a_j-b_i-1);\ j=i+1,i+2,\ldots,r\}$$
$$\{(N+a_i+a_j);\ j=r+1,r+2,\ldots,\tilde{\lambda}_i\} \qquad \{(N-b_i-b_j-2);\ j=r+1,r+2,\ldots,\lambda_i\} \tag{3.23}$$

for each fixed value of i in the range $1\leqslant i\leqslant r$. Moreover

$$\prod_{i=1}^{r}\frac{(N+a_i-1)!}{(N-b_i-2)!}=\prod_{j=1}^{a_i+b_i+1}(N+a_i-j) \tag{3.24}$$

and the set

$$\{(N+a_i-j):j=1,2,\ldots,a_i+b_i+1\} \tag{3.25}$$

is precisely the disjoint union of the four sets (3.23). The inclusion of the first two sets in (3.25) corresponds to the claim that the cancellations referred to do in fact take place. They are consequences of the inequalities

$$0\leqslant b_r<b_{r-1}<\ldots<b_i\leqslant a_i+b_i$$

and

$$0\leqslant a_r<a_{r-1}<\ldots<a_{i+1}<a_i\leqslant a_i+b_i.$$

The inclusion of the second two sets of (3.23) in (3.25) follow from the inequalities

$$-1\geqslant a_{r+1}>a_{r+2}>\ldots>a_{\tilde{\lambda}_i}=\lambda_{\tilde{\lambda}_i}-\tilde{\lambda}_i\geqslant i-\tilde{\lambda}_i=-b_i>-a_i-b_i-1$$

since $\lambda_{\tilde{\lambda}_i}\geqslant i$, and

$$-1\geqslant b_{r+1}>b_{r+2}>\ldots>b_{\lambda_i}=\tilde{\lambda}_{\lambda_i}-\lambda_i\geqslant i-\lambda_i=-a_i>-a_i-b_i-1$$

since $\tilde{\lambda}_{\lambda_i}\geqslant i$. Having proved the necessary inclusions, the fact that equation (3.25) is the union of the disjoint sets (3.23) follows from the fact that the total number of elements in these four subsets is $(r-i+1)+(r-i)+(\tilde{\lambda}_i-r)+(\lambda_i-r)$ which is simply $\lambda_i-i+\tilde{\lambda}_i-i+1=a_i+b_i+1$, the number of elements in (3.25). Thus, cancelling and collecting together similar terms yields

$$D_N\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}=\prod_{i=1}^{r}\left(\prod_{j=i}^{\tilde{\lambda}_i}(N+a_i+a_j)\prod_{j=i+1}^{\lambda_i}(N-b_i-b_j-2)\right)\Big/H\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix} \tag{3.26}$$

and

$$D_N\left\langle\begin{matrix}\boldsymbol{a}\\ \boldsymbol{b}\end{matrix}\right\rangle=\prod_{i=1}^{r}\left\{\prod_{j=i+1}^{\tilde{\lambda}_i}(N+a_i+a_j+2)\prod_{j=1}^{\lambda_i}(N-b_i-b_j)\right\}\Big/H\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}. \tag{3.27}$$

In terms of the partition labels $\boldsymbol{\lambda}$:

$$D_N[\boldsymbol{\lambda}]=\prod_{(i\geqslant j)}^{\boldsymbol{\lambda}}(N+\lambda_i+\lambda_j-i-j)\prod_{(i<j)}^{\boldsymbol{\lambda}}(N-\tilde{\lambda}_i-\tilde{\lambda}_j+i+j-2)\Big/H(\boldsymbol{\lambda}) \tag{3.28}$$

and

$$D_N\langle\boldsymbol{\lambda}\rangle=\prod_{(i>j)}^{\boldsymbol{\lambda}}(N+\lambda_i+\lambda_j-i-j+2)\prod_{(i\leqslant j)}^{\boldsymbol{\lambda}}(N-\tilde{\lambda}_i-\tilde{\lambda}_j+i+j)\Big/H(\boldsymbol{\lambda}) \tag{3.29}$$

where the products are taken over all pairs (i,j) specifying positions of boxes of the Young diagrams corresponding to the partition $\boldsymbol{\lambda}$ with the indicated restrictions on i and j for each particular product.

In these forms it is easy to write down the numerators of these expressions as factored polynomials by entering a number in each box of the Young diagram specified by $\boldsymbol{\lambda}$. The resulting array is composed of two regions, above and below the main diagonal, with the boxes on the main diagonal included in the lower and upper regions for the orthogonal and symplectic groups respectively. The entries themselves consist of one or other of the factors $(N-i-j)$, $(N+i+j-2)$, $(N-i-j+2)$ or $(N+i+j)$ with either $\lambda_i+\lambda_j$ added or $\tilde{\lambda}_i+\tilde{\lambda}_j$ subtracted. In the case of the partition $\boldsymbol{\lambda}=(4\ 3^2\ 2\ 1)$ this is exemplified by the arrangements

$$
\begin{array}{c|cccc|cl}
 & -5 & -4 & -3 & -1 & -\tilde{\boldsymbol{\lambda}} & \to i \\
\hline
4 & N-2 & N+1 & N+2 & N+3 & -5 & \\
3 & N-3 & N-4 & N+3 & & -4 & \\
3 & N-4 & N-5 & N-6 & i<j & -3 & \\
2 & N-5 & N-6 & & i\geqslant j & -1 & \\
1 & N-6 & & & & & \\
\hline
+\boldsymbol{\lambda} & 4 & 3 & 3 & 2 & & \\
\downarrow j & & & & & &
\end{array}
$$

and

$$
\begin{array}{c|ccccc|cl}
 & -5 & -4 & -3 & & -1 & -\tilde{\boldsymbol{\lambda}} & \to i \\
\hline
4 & N+2 & N+3 & N+4 & & N+5 & -5 & \\
3 & N-1 & N+4 & N+5 & & & -4 & \\
3 & N-2 & N-3 & N+6 & & & -3 & \\
2 & N-3 & N-4 & & i\leqslant j & & -1 & \\
1 & N-4 & & & i>j & & & \\
\hline
+\boldsymbol{\lambda} & 4 & 3 & 3 & & 2 & & \\
\downarrow j & & & & & & &
\end{array}
$$

Thus

$$
D_N[4\ 3^2\ 2\ 1]=\begin{array}{llll}
(N+6) & (N-8) & (N-6) & (N-3) \\
(N+4) & (N+2) & (N-4) & \\
(N+3) & (N+1) & (N) & \\
(N+1) & (N-1) & & \\
(N-1) & & &
\end{array}\Bigg/\begin{array}{llll}
8 & 6 & 4 & 1 \\
6 & 4 & 2 & \\
5 & 3 & 1 & \\
3 & 1 & & \\
1 & & &
\end{array}
$$

and

$$
D_N\langle 4\ 3^2\ 2\ 1\rangle=\begin{array}{llll}
(N-8) & (N-6) & (N-4) & (N-1) \\
(N+6) & (N-4) & (N-2) & \\
(N+5) & (N+3) & (N) & \\
(N+3) & (N+1) & & \\
(N+1) & & &
\end{array}\Bigg/\begin{array}{llll}
8 & 6 & 4 & 1 \\
6 & 4 & 2 & \\
5 & 3 & 1 & \\
3 & 1 & & \\
1 & & &
\end{array}
$$

It should be noted that for the special case $N=2k=2\tilde{\lambda}_1=10$ the dimensions of the appropriate irreducible representations of SO(10) are given by

$$D_{10}[4\ 3^2\ 2\ 1]_{\pm}=\tfrac{1}{2}\ .\ \begin{array}{cccccccccc} 16 & 2 & 4 & 7 & / & 8 & 6 & 4 & 1 \\ 14 & 12 & 6 & & / & 6 & 4 & 2 & \\ 13 & 11 & 10 & & / & 5 & 3 & 1 & \\ 11 & 9 & & & / & 3 & 1 & & \\ 9 & & & & / & 1 & & & \end{array}$$

$$=13\ 11^2\ 7^2\ 3^2\ 2.$$

4. Composite Young diagrams

In extending the results of the previous section to the irreducible representations of U(N) whose characters $\{\bar{\nu};\boldsymbol{\mu}\}$ are associated with composite Young diagrams it is convenient to take as the starting point the expansion obtained earlier (El Samra and King 1979):

$$\{\bar{\nu};\boldsymbol{\mu}\}=|\{1^{\overline{d_j+m}};a_i+m\}| \tag{4.1}$$

where $a_i=\mu_i-i$, $b_j=\tilde{\mu}_j-j$, $c_i=\nu_i-i$, and $d_j=\tilde{\nu}_j-j$ for $i,j=1,2,\ldots,m$ with $m\geqslant\mu_1$, $m\geqslant\tilde{\mu}_1$, $m\geqslant\nu_1$ and $m\geqslant\tilde{\nu}_1$.

The relation

$$\{\bar{\nu};\boldsymbol{\mu}\}=\bar{\epsilon}^m\{\boldsymbol{\lambda}\} \tag{4.2}$$

for characters of U(N) with

$$\tilde{\lambda}_j=\begin{cases} N-\tilde{\nu}_{m-j+1} & \text{for } j=1,2,\ldots,m \\ \tilde{\mu}_{j-m} & \text{for } j=m+1,m+2,\ldots,2m \end{cases}$$

is such that

$$\{1^{\overline{d+m}};a+m\}=\bar{\epsilon}\{a+m+1,1^{N-d-m-1}\}, \tag{4.3}$$

so that from (2.7)

$$\begin{aligned} D_N\{1^{\overline{d+m}};a+m\}&=D_N\{a+m+1,1^{N-d-m-1}\} \\ &=(N+a+m)!/(d+m)!(a+m)!(N-d-m-1)!(N+a-d). \end{aligned} \tag{4.4}$$

Using this result in combination with (4.1) gives, after extracting the factors common to each element in any given row or column, the result

$$D_N\{\bar{\nu};\boldsymbol{\mu}\}=\prod_{i=1}^{m}\frac{(N+a_i+m)!}{(N-d_i-m-1)!(a_i+m)!(d_i+m)!}\left|\frac{1}{(N+a_i-d_j)}\right|. \tag{4.5}$$

Recourse, once again, to Cauchy's lemma gives

$$\begin{aligned} D_N\{\bar{\nu};\boldsymbol{\mu}\}=&\prod_{i=1}^{m}\frac{(N+a_i+m)!}{(N-d_i-m-1)!(a_i+m)!(d_1+m)!} \\ &\times\prod_{1\leqslant i<j\leqslant m}(a_i-a_j)(d_i-d_j)\left(\prod_{i=1}^{m}\prod_{j=1}^{m}(N+a_i-d_j)\right)^{-1}. \end{aligned} \tag{4.6}$$

However, from Frobenius' lemma, for each $i = 1, 2, \ldots, m$ the sets

$$\{(a_i - a_j): j = i+1, i+2, \ldots, m\} \quad \text{and} \quad \{(a_i + b_j + 1): j = 1, 2, \ldots, \mu_i\}$$

are disjoint. Their union is seen to be

$$\{(a_i + m + 1 - j): j = 1, 2, \ldots, a_i + m\}.$$

The relevant inclusions are demonstrated by noting that

$$a_i > a_{i+1} > a_{i+2} > \ldots > a_m = \mu_m - m \geqslant -m$$

and

$$m \geqslant \tilde{\mu}_1 = b_1 + 1 > b_1 > b_2 > \ldots > b_{\mu_i} = \tilde{\mu}_{\mu_i} - \mu_i \geqslant i - \mu_i = -a_i.$$

Thus

$$\frac{1}{(a_i+m)!} \prod_{j=i+1}^{m} (a_i - a_j) = \left(\prod_{j=1}^{\mu_i} (a_i + b_j + 1) \right)^{-1}. \tag{4.7}$$

Similarly

$$\frac{1}{(b_i+m)!} \prod_{j=i+1}^{m} (d_i - d_j) = \left(\prod_{j=1}^{\tilde{\nu}_i} (c_j + d_i + 1) \right)^{-1}, \tag{4.8}$$

so that

$$D_N\{\tilde{\boldsymbol{\nu}}; \boldsymbol{\mu}\} = \prod_{i=1}^{m} \frac{(N+a_i+m)!}{(N+a_i-d_i-1)!} \left(\prod_{j=1}^{m} (N + a_i - d_j) \right)^{-1}$$
$$\times \prod_{i=1}^{m} \frac{(N+a_i-d_i-1)!}{(N-d_i-m-1)!} \left(\prod_{j=i+1}^{m} (N + a_j - d_i) \right)^{-1} (H(\boldsymbol{\mu})H(\boldsymbol{\nu}))^{-1}. \tag{4.9}$$

Proceeding in a now familiar manner, Frobenius' lemma is such that for each $i = 1, 2, \ldots, m$ the sets

$$\{(N + a_i - d_j): j = i, i+1, \ldots, m\} \quad \text{and} \quad \{(N + a_i + c_j + 1): j = 1, 2, \ldots, \nu_i\}$$

are disjoint. They are both subsets of

$$\{(N + a_i + m + 1 - j): j = 1, 2, \ldots, m + d_i + 1\}$$

since

$$d_i > d_{i+1} > d_{i+2} > \ldots > d_m = \nu_m - m \geqslant -m$$

and

$$m \geqslant \nu_1 = c_1 + 1 > c_1 > c_2 > \ldots > c_{\nu_i} = \tilde{\nu}_{\nu_i} - \nu_i \geqslant i - \nu_i = -d_i..$$

Counting the elements then shows that the union of the two subsets is the containing set. Similarly the two sets

$$\{(N - d_i + a_j): j = i+1, i+2, \ldots, m\} \qquad \text{and} \qquad \{(N - d_i - b_j - 1): j = 1, 2, \ldots, \mu_i\}$$

are disjoint and their union is

$$\{(N - d_i + a_i - j): j = 1, 2, \ldots, m + a_i\}.$$

These observations lead to the required simplifications of (4.9):

$$D_N\{\bar{\boldsymbol{\nu}};\boldsymbol{\mu}\}=\left(\prod_{i=1}^{m}\prod_{j=1}^{\tilde{\nu}_i}(N+a_i+c_j+1)\right)\left(\prod_{i=1}^{m}\prod_{j=1}^{\mu_i}(N-d_i-b_j-1)\right)\Big/H(\boldsymbol{\mu})H(\boldsymbol{\nu}) \qquad (4.10)$$

which in terms of the partition labels $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ is simply

$$D_N\{\bar{\boldsymbol{\nu}};\boldsymbol{\mu}\}=\left(\prod_{(i,j)}^{\boldsymbol{\mu}}(N-\tilde{\nu}_i-\tilde{\mu}_j+i+j-1)/H(\boldsymbol{\mu})\right)\left(\prod_{(k,l)}^{\boldsymbol{\nu}}(N+\nu_k+\mu_l-k-l+1)/H(\boldsymbol{\nu})\right).$$

This result was derived earlier in a much more laborious way (El Samra 1970, Jahn and El Samra 1970 (unpublished)). It is in a form which makes it very easy to write down the result as a product of two arrays of N-dependent factors divided by products of hook lengths appropriate to the Young diagrams specified by $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ which constitute the composite Young diagram associated with $\{\bar{\boldsymbol{\nu}};\boldsymbol{\mu}\}$. The arrays are formed by adding the row lengths $\nu_k+\mu_l$ to and subtracting the column lengths $\tilde{\nu}_i+\tilde{\mu}_j$ from the basic entries of the form $(N-k-l+1)$ and $(N+i+j-1)$ in the dotted boxes specified by $\boldsymbol{\nu}$ and the boxes specified by $\boldsymbol{\mu}$, respectively. This is illustrated in the case $\{\bar{\boldsymbol{\nu}};\boldsymbol{\mu}\}=\{\overline{4\,3\,1};2^2\,1\}$ by the arrangement

$$\begin{array}{c|cccc|cc|c}
{}_{\boldsymbol{\nu}}\diagdown^{\boldsymbol{\mu}} & 0 & 1 & 2 & 2 & -3 & -2 & {}^{-\tilde{\boldsymbol{\mu}}}\diagup_{-\tilde{\boldsymbol{\nu}}} \\
l\leftarrow & & & & & & & \rightarrow j \\
\hline
4 & N-4 & N-3 & N-2 & N-1 & N+1 & N+2 & -3 \\
3 & & N-4 & N-3 & N-2 & N+2 & N+3 & -2 \\
1 & & & & N-3 & N+3 & & -2 \\
 & & & & & & & -1 \\
 & & & & k & \downarrow\ i & &
\end{array}$$

Hence

$$D_N\{\overline{4\,3\,1};2^2\,1\}=\frac{\begin{array}{cccc} N & (N+2) & (N+4) & (N+5) \\ & N & (N+2) & (N+3) \\ & & & N \end{array}}{\begin{array}{cccc} 1 & 3 & 4 & 6 \\ & 1 & 2 & 4 \\ & & & 1 \end{array}}\cdot\frac{\begin{array}{cc} (N-5) & (N-3) \\ (N-3) & (N-1) \\ (N-2) & \end{array}}{\begin{array}{cc} 4 & 2 \\ 3 & 1 \\ 1 & \end{array}}.$$

5. Spinor characters

It is only necessary to complete the discussion with an account of the irreducible spinor representations of O(N). Such representations have characters denoted by $[\boldsymbol{\Delta};\boldsymbol{\lambda}]$ or to be more explicit $[\boldsymbol{\Delta};\boldsymbol{\lambda}]_N$. In the preceding paper (El Samra and King 1979) it has been shown that for certain group elements including the identity element of the group

$$[\boldsymbol{\Delta};\boldsymbol{\lambda}]_N=\boldsymbol{\Delta}_N\langle\boldsymbol{\lambda}\rangle_{N-1}. \qquad (5.1)$$

Here $\boldsymbol{\Delta}_N$ denotes the basic spin representation character of dimension 2^k for both $N=2k+1$ and $N=2k$ and $\langle\boldsymbol{\lambda}\rangle_{N-1}$ is the character of the irreducible representation of

Sp$(2k)$ if $N=2k+1$ and is defined formally for all N by

$$\langle \boldsymbol{\lambda} \rangle_{N-1} = |\{\lambda_i - i + j\}_{N-1} + (1-\delta_{1j})\{\lambda_i - i - j + 2\}_{N-1}|, \tag{5.2}$$

where the complete homogeneous symmetric functions $\{m\}$ are defined with respect to $N-1$ eigenvalues of the group element of O(N) under consideration, excluding one whose value is 1.

In the case of the identity element of the group, (5.1) yields, for $N=2k+1$ and $N=2k$, the result

$$D_N[\boldsymbol{\Delta}; \boldsymbol{\lambda}] = 2^k D_{N-1}\langle \boldsymbol{\lambda} \rangle$$

where by virtue of the validity of (5.2) for all N, $D_{N-1}\langle \boldsymbol{\lambda} \rangle$ may be obtained from (3.29) merely by replacing N by $N-1$. Therefore

$$D_N[\boldsymbol{\Delta}; \boldsymbol{\lambda}] = 2^k \prod_{(i>j)}^{\boldsymbol{\lambda}} (N+\lambda_i+\lambda_j-i-j+1) \prod_{(i\leqslant j)}^{\boldsymbol{\lambda}} (N-\tilde{\lambda}_i-\tilde{\lambda}_j+i+j-1)/H(\boldsymbol{\lambda}) \tag{5.3}$$

for $N=2k+1$ and $N=2k$. Thus for example

$$D_N[\boldsymbol{\Delta}; 4\,3^2\,2\,1] = \begin{array}{llll} (N-9) & (N-7) & (N-5) & (N-2)\ 2^k \\ (N+5) & (N-5) & (N-3) & \\ (N+4) & (N+2) & (N+1) & \\ (N+2) & N & & \\ N & & & \end{array} \Bigg/ \begin{array}{llll} 8 & 6 & 4 & 1 \\ 6 & 4 & 2 & \\ 5 & 3 & 1 & \\ 3 & 1 & & \\ 1 & & & \end{array}$$

and in the case of SO$(2k)$:

$$D_{2k}[\boldsymbol{\Delta}; 4\,3^2\,2\,1]_\pm = \tfrac{1}{2} D_{2k}[\boldsymbol{\Delta}; 4\,3^2\,2\,1].$$

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